Quiz 5 Solutions

1. Use Stokes' Theorem to evaluate $\int_C F \cdot dr$, where $F(x, y, z) = x^2y\, i + \frac{x^3}{3}\, j + xy\, k$ and C is the curve of intersection of $z = y^2 - x^2$ and $x^2 + y^2 = 1$ oriented counterclockwise as viewed from above.

   Solution. Let S be part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies above the unit disk D centered at origin. Clearly, S is a smooth surface and C is a smooth curve. Also, all the components of F are polynomials in x and y, which have continuous first partials everywhere in $\mathbb{R}^3$. Therefore, the hypothesis of Stokes' Theorem are satisfied.

   By a simple calculation, we have that $\nabla \times F = xi - yj$ and the unit normal to the surface is $n = \frac{2xi - 2yj + k}{\sqrt{4x^2 + 4y^2 + 1}}$. By Stokes' Theorem, we have that

$$\begin{aligned}
\int_C F \cdot dr &= \iint_S (\nabla \times F) \cdot n\, d\sigma \\
&= \iint_D (\nabla \times F) \cdot n \sqrt{f_x^2 + f_y^2 + 1}\, dA \\
&\text{(Here z } = f(x, y) = y^2 - x^2) \\
&= 2 \iint_D (x^2 + y^2)\, dA \\
&= 4 \int_0^{2\pi} \int_0^r r^3\, dr\, d\theta \\
&= \pi.
\end{aligned}$$

2. Verify the Gauss' Divergence Theorem for the vector field $F(x, y, z) = xy\, i + yz\, j + zx\, k$ on the region $E:\ x^2 + y^2 \leq 1,\ 0 \leq z \leq 1$.

   Solution. As in Problem 1, here too the components of F are polynomials, and hence have continuous first partials throughout $\mathbb{R}^3$. Furthermore, ∂E comprises three components: the two disks $S_1 : x^2 + y^2 = 1, z = 1$, $S_2 : x^2 + y^2 = 1, z = 0$, and the lateral surface $S_3 : x^2 + y^2 = 1, 0 \leq z \leq 1$. Since ∂E is a piecewise smooth surface, the hypotheses of Gauss' Divergence Theorem are satisfied. Therefore, by the Gauss' Divergence Theorem, we have that

$$\int_{\partial E} F \cdot n\, d\sigma = \iint_E (\nabla \cdot F)\, dV,$$

which needs to be verified.

By a simple calcuilation, we can see that $\nabla \cdot F = x + y + z$. Hence, we have that

$$\begin{aligned}
\iiint_E (\nabla \cdot F)\, dV &= \iiint_E (x + y + z)\, dV \\
&= \int_0^{2\pi}\int_0^1\int_0^1 (r\cos\theta + r\sin\theta + z)\, r\, dz\, dr\, d\theta \\
&= \pi/2.
\end{aligned}$$

Since $\partial E = S_1 \sqcup S_2 \sqcup S_3$, we have that

$$\iint_{\partial E} F \cdot n\, d\sigma = \iint_{S_1} F \cdot n\, d\sigma + \iint_{S_2} F \cdot n\, d\sigma + \iint_{S_3} F \cdot n\, d\sigma.$$

It is easy to see that $n = k$ and $n = -k$ for S_1 and S_2, respectively. So

$$\begin{aligned}
\iint_{S_1} F \cdot n\, d\sigma &= \iint_{S_1} F \cdot k\, d\sigma \\
&= \iint_{S_1} x\, d\sigma \\
&= \int_0^{2\pi}\int_0^1 (r\cos\theta)\, r\, dr\, d\theta \\
&= 0.
\end{aligned}$$

In a similar fashion, we can conclude that $\iint_{S_2} F \cdot n\, d\sigma = 0$.

On S_3, $n = \frac{2xi+2yj}{4x^2+y^2} = xi + yj$, and we can parametrize S_3 by $R(r, \theta, z) = \cos\theta i + \sin\theta j + zk$, $r \in [0, 1]$ and $\theta \in [0, 2\pi]$. Hence, we have that

$$\begin{aligned}
\iint_{S_3} F \cdot n\, d\sigma &= \iint_{S_3} (x^2 y + y^2 z)\, d\sigma \\
&= \int_0^{2\pi}\int_0^1 (\cos^2\theta \sin\theta + z\sin^2\theta)|R_\theta \times R_z|\, dz\, d\theta \\
&= \pi/2.
\end{aligned}$$

We conclude from these calculations that

$$\int_{\partial E} F \cdot n\, d\sigma = \iint_E (\nabla \cdot F)\, dV,$$

which verifies the Gauss' Divergence Theorem.

3. Solve the differential equation

$$(y^2 - 3xy - 2x^2)\,dx + (xy - x^2)\,dy = 0.$$

Solution. Since $P_y = 2y - 3x$ is not equal to $Q_x = y - 2x$, so we have that the equation is not exact. To find the integrating factor, we use the form

$$F(x) = \frac{P_y - Q_x}{Q} = \frac{y - x}{x(y - x)} = \frac{1}{x}, \text{ if } y \neq x.$$

Therefore, the integrating factor is

$$h(x) = e^{\int F(x)\,dx} = x.$$

Multiplying the equation by this factor, we obtain the exact equation

$$(xy^2 - 3x^2y - 2x^3)\,dx + (x^2y - x^3)\,dy = 0.$$

A solution to this equation is given by

$$f(x, y) = \int_0^x P(x, y)\,dx + \int_0^y Q(0, y)\,dy = c,$$

by taking $(x_0, y_0) = (0, 0)$, where both P and Q are defined. Therefore, we have that $\frac{x^2y^2}{2} - x^3y - \frac{x^4}{2} = c$ is a solution.